CHAPTER IV

Multiple Integrals

The idea of differentiation and the operations with derivatives in the case of functions of **several variables** are obtained almost immediately by reduction to their analogues for functions of **one variable**. On the other hand, as regards integration and its relation to differentiation, the case of several variables is more involved, since the concept of the integral can be generalized for functions of several variables in a variety of ways. In this chapter, we shall study the **multiple integrals** which we have already wencountered. However, in addition, we must also consider the so-called **line integrals in the plane**, **surface integrals** and **line integrals in three dimensions** (Chapter V). However, in the end, we will discover that all questions of integration can be reduced to the original concept of the integral in the case of one independent variable.

4.1 ORDINARY INTEGRALS AS FUNCTIONS OF A PARAMETER

Before we study the new situations which arise with functions of more than one variable, we shall discuss some **concepts** which are directly related to matters already familiar to us.

4.1.1 Examples and Definitions: If $f(x, y)$ is a **continuous function** of x, y in the rectangular region $\alpha \le x \le \beta$, $a \le y \le b$, we may, in the first instance, think of the quantity x as fixed andcan then integrate the function $f(x, y)$ - now a function of y alone - over the interval $a \le y \le b$. We thus arrive at the expression

$$\int_a^b f(x, y)\,dy,$$

which still depends on the choice of the quantity x. Hence, in a sense, we are not considering an integral, but a family of integrals $\int_a^b f(x, y)\,dy$ which we obtain for different values x. This quantity, which is kept fixed during the integration and to which we can assign any value in its interval, we call a **parameter**. . Hence, our **ordinary integral** appears as **function of the parameter** x.

Integrals, which are functions of a parameter, frequently occur in **analysis and its applications**.

Thus, as the substitution $xy = u$ readily shows,

$$\int_0^1 \frac{x\,dy}{\sqrt{(1 - x^2y^2)}} = \text{arsin}\,x.$$

Again, while integrating the **general power function**, we may regard the index as a **parameter** and accordingly write

$$\int_0^1 y^x\,dy = \frac{1}{x+1},$$

where we may assume that $x > -1$.

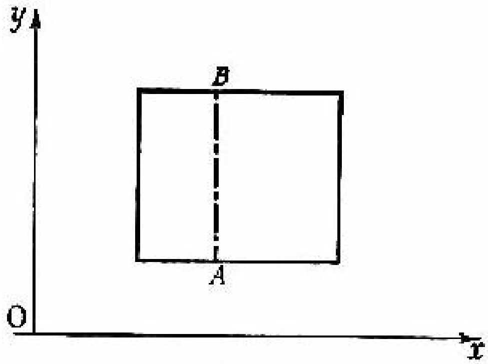

Fig. 1

If we represent the region of definition of the function $f(x, y)$ geometrically and let the parallel to the y-axis corresponding to the fixed value of x intersect the rectangle as in Fig. 1, we then obtain the function of y which is to be integrated by considering the values of the function $f(x, y)$ as a function of y along the segment AB. We may also speak of integrating the function $f(x, y)$ along the segment AB.

This **geometrical point of view** suggests a generalization. If the region of definition R, in which the function $f(x, y)$ is considered, is not a rectangle, but has instead the shape shown in Fig. 2 (i.e., if any parallel to the y-axis intersects the boundary in at most two points), then, we canintegrate for a fixed value of x the values of the function $f(x, y)$ along the line AB in which the parallel to the y-axis intersects the region of definition R. The initial and final points of the interval of integration will themselves vary with x. In other words, we have to consider an integral of the type

$$\int_{\psi_1(x)}^{\psi_2(x)} f(x, y)\, dy = F(x),$$

i.e., an integral with the variable of integration y and the parameter x, in which the parameter occurs in the **integrand** as well as in the **limits of integration**.

For example, if the region of definition is a circle with unit radius and centre at the origin, we shall have to consider integrals of the type

$$\int_{-\sqrt{(1-x^2)}}^{+\sqrt{(1-x^2)}} f(x, y)\, dy.$$

4.1.2. Continuity and Differentiability of an Integral with respect to the Parameter: The integral

$$F(x) = \int_a^b f(x, y)\, dy$$

is a continuous function of the parameter x, if $f(x, y)$ is continuous in the region in question.

In fact,

$$\left|F(x+h) - F(x)\right| = \left|\int_a^b (f(x+h, y) - f(x, y))\, dy\right|$$
$$\leq \int_a^b \left|f(x+h, y) - f(x, y)\right| dy.$$

By virtue of the **uniform continuity** of $f(x, y)$ for sufficiently small values of h, the integrand on the right hand side, considered as a function of y, may be made uniformly as small as we please, and the statement follows immediately. In particular, therefore, we can integrate the function $F(x)$ with respect to the parameter x between the limits α and β, obtaining

$$\int_\alpha^\beta F(x)\, dx = \int_\alpha^\beta \left(\int_a^b f(x, y)\, dy\right) dx.$$

We can also write the integral on the right hand side in the form

$$\int_\alpha^\beta \int_a^b f(x, y)\, dy\, dx;$$

we call it a **repeated or multiple integral** (in this case also a **double integral**).

We will now investigate the possibility of **differentiating** $F(x)$. In the first place, we consider the case when the limits are fixed and assume that the function $f(x, y)$ has a continuous partial derivative f_x throughout the closed rectangle R. It is natural to try to form the x-derivative of the integral in the following way: Instead of first integrating and then differentiating, we **reverse** the order of these two processes, i.e., we **first differentiate** with respect to x and **then integrate** with respect to y. As a matter of fact, the following **theorem** is true:

If in the closed rectangle $\alpha \le x \le \beta$, $a \le y \le b$ the function $f(x, y)$ has a continuous derivative with respect to x, we may differentiate the integral with respect to the parameter under the integral sign, i.e., if $\alpha \le x \le \beta$,

$$\frac{d}{dx} F(x) = \frac{d}{dx} \int_a^b f(x, y)\, dy = \int_a^b f_x(x, y)\, dy.$$

We thus obtain a simple proof of the fact, which we have already proved , that, in the formation of the mixed derivative g_{xy} of a function $g(x, y)$, the order of differentiation can be changed provided that g_{xy} is continuous and g_x exists. In fact, if we set $f(x, y) = g_y(x, y)$, we have

$$g(x, y) = g(x, a) + \int_a^y f(x, \eta)\, d\eta.$$

Since $f(x, y)$ has a continuous derivative with respect to x in the rectangle $\alpha \le x \le \beta$, $a \le y \le b$, it follows that

$$g_x = g_x(x, a) + \int_a^y f_x(x, \eta)\, d\eta,$$

Proof of theorem: If both x and $x + h$ belong to the interval $\alpha \le x \le \beta$, we can write

$$F(x+h) - F(x) = \int_a^b f(x+h, y)\, dy - \int_a^b f(x, y)\, dy$$

$$= \int_a^b \{f(x+h, y) - f(x, y)\}\, dy.$$

Since we have assumed that $f(x, y)$ is differentiable, the **Mean value theorem of differential calculus in its usual form** yields

$$f(x+h, y) - f(x, y) = hf_x(x + \theta h, y), \quad 0 < \theta < 1.$$

Here the quantity θ depends on y, and may even vary **discontinuously** with y. This does not matter, because we see at once from the equation

$$f_x(x + \theta h, y) = h^{-1}(f(x + h, y) - f(x, y))$$

that $f_x(x + \theta h, y)$ is a **continuous** function of x and y, and is therefore **integrable**.

Moreover, since the derivative f_x is assumed to be continuous in the closed region and therefore **uniformly continuous**, the absolute value of the difference

$$f_x(x + \theta h, y) - f_x(x, y)$$

is less than a positive quantity ε which is independent of x and y and tends to zero with h. Thus

$$\left| \frac{F(x+h) - F(x)}{h} - \int_a^b f_x(x, y)\, dy \right|$$

$$= \left| \int_a^b f_x(x + \theta h, y)\, dy - \int_a^b f_x(x, y)\, dy \right| \le \int_a^b \epsilon\, dy = \epsilon(b - a).$$

If we now let h tend to zero, ε also tends to zero and the relation

$$\lim_{h \to 0} \frac{F(x+h) - F(x)}{h} = \int_a^b f_x(x, y)\, dy = F'(x)$$

follows at once; our statement is thus proved.

In a similar way, we can establish the **continuity** of the integral and the **rule for differentiating** the integral with respect to a parameter when the parameter occurs in the **limits**. For example, if we wish to differentiate

$$F(x)=\int_{\psi_1(x)}^{\psi_2(x)} f(x, y)\, dy,$$

we start with

$$F(x)=\int_u^v f(x, y)\, dy=\Phi(u, v, x),$$

where $u = \psi_1(x)$, $v = \psi_2(x)$. We assume here that $\psi_1(x)$ and $\psi_2(x)$ have continuous derivatives with respect to x throughout the interval and that $f(x, y)$ is continuously differentiable (2.4.2) in a region wholly enclosing the region R. By the Chain Rule, we now obtain

$$F'(x)=\frac{\partial \Phi}{\partial x}+\frac{\partial \Phi}{\partial u} \frac{d u}{d x}+\frac{\partial \Phi}{\partial v} \frac{d v}{d x}.$$

If we apply the **fundamental theorem of the integral calculus**, this yields

$$F'(x)=\int_{\psi_1(x)}^{\psi_2(x)} f_x(x, y)\, dy-\psi_1'(x) f(x, \psi_1(x))+\psi_2'(x) f(x, \psi_2(x)).$$

Thus, if we take for $F(z)$ the function

$$F(x)=\int_0^x \sin(x y)\, dy,$$

we obtain

$$\frac{d F(x)}{d x}=\int_0^x y \cos(x y)\, dy+\sin(x^2).$$

If we take

$$F(x)=\int_0^1 \frac{x\, dy}{\sqrt{(1-x^2 y^2)}}=\operatorname{arsin} x,$$

as the reader will *verify* directly.

Other examples are given by the integrals

$$F_n(x)=\int_0^x \frac{(x-y)^n}{n!} f(y)\, dy, \quad F_0(x)=\int_0^x f(y)\, dy,$$

where n is any positive integer and $f(y)$ is a continuous function of y only in the interval under consideration. Since the expression arising from differentiation with respect to the upper limit x vanishes, the rule yields

$$F_n'(x) = F_{n-1}(x).$$

Since $F_0'(x) = f(x)$, this yields at once

$$F_n^{(n+1)}(x) = f(x).$$

Hence $F_n(x)$ is the function the $(n + 1)$-th derivative of which equals $f(x)$ and which, together with its first n derivatives, vanishes when $x = 0$; it arises from $F_{n-1}(x)$ by integration from 0 to x. Hence.$F_n(x)$ is the function which is obtained from $f(x)$ by integrating n+1 times between the limits 0 and x. This repeated integration can therefore be replaced by a single integration of the function $[(x - y)^n f(y)]/n!$ with respect to y.

The rules for **differentiating an integral with respect to a parameter** often remain valid even when differentiation under the integral sign gives a function which is not continuous everywhere. In such cases, instead of applying general criteria, it is more convenient to verify in each special case whether such a differentiation is permissible.

As an example, consider the elliptic integral

$$F(k) = \int_{-1}^{+1} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}; \quad (k^2 < 1).$$

The function

$$f(k, x) = \frac{1}{\sqrt{(1-x^2)(1-k^2x^2)}}$$

is discontinuous at $x = +1$ and at $x = -1$, but the integral (as an **improper integral**) has a meaning. Formal differentiation with respect to the parameter k yields

$$F'(k) = \int_{-1}^{+1} \frac{kx^2\,dx}{\sqrt{(1-x^2)(1-k^2x^2)^3}}.$$

In order to discover whether this equation is correct, we repeat the argument by which we obtained our differentiation formula. This yields

$$\frac{F(k+h)-F(k)}{h} = \int_{-1}^{+1} f_k(k+\theta h, x)\,dx = \int_{-1}^{+1} \frac{(k+\theta h)x^2\,dx}{\sqrt{(1-x^2)(1-(k+\theta h)^2x^2)^3}}.$$

The difference between this expression and the integral obtained by formal differentiation is

$$\Delta = \int_{-1}^{+1} \frac{x^2}{\sqrt{1-x^2}} \left(\frac{k+\theta h}{\sqrt{(1-(k+\theta h)^2 x^2)^3}} - \frac{k}{\sqrt{(1-k^2x^2)^3}} \right) dx.$$

We must show that this integral tends to zero with h. For this purpose, we mark off about k an interval $k_0 \leq k \leq k_1$ which does not contain the values ± 1 and choose h so small that $k + \theta h$ lies in this interval. The function

$$\frac{k}{\sqrt{(1-k^2x^2)^3}}$$

is continuous in the closed region $-1 \leq \xi \leq 1$, $k_0 \leq k \leq k_1$ and is therefore **uniformly continuous**. Consequently, the difference

$$\left| \frac{k+\theta h}{\sqrt{(1-(k+\theta h)^2 x^2)^3}} - \frac{k}{\sqrt{(1-k^2x^2)^3}} \right|$$

remains below a bound ε which is independent of x and k, and tends to zero with h. Hence, the absolute value of the integral Δ also remains less than

$$\int_{-1}^{+1} \frac{x^2\,dx}{\sqrt{1-x^2}}\,\varepsilon = M\varepsilon,$$

where M is a constant independent of ε, i.e., the integral Δ tends to zero with h, which is what we wanted to show.

Hence, **differentiation under the integral sign** is permissible in this case. Similar considerations lead to the required result in other cases.

Improper integrals with an infinite range of integration are discussed in the A4.4.1

Exercises 4.1

1. Evaluate

$$F(y) = \int_0^1 x^{y-1}(y \log x + 1)\,dx.$$

2. Let $f(x, y)$ be twice continuously differentiable and $u(x, y, z)$ defined as follows:

$$u(x, y, z) = \int_0^{2\pi} f(x + z\cos\varphi,\, y + z\sin\varphi)\,d\varphi.$$

Prove that

$$z(u_{xx} + u_{yy} - u_{zz}) - u_z = 0.$$

3 *. If $f(x)$ is twice continuously differentiable and

$$u(x, t) = \frac{1}{t^{p-2}} \int_{-t}^{+t} f(x+y)\,(t^2 - y^2)^{\frac{p-3}{2}}\, dy \qquad (p > 1),$$

prove that

$$u_{xx} = \frac{p-1}{t} u_t + u_{tt}.$$

4. The **Bessel function $J_0(x)$** may be defined by

$$J_0(x) = \frac{1}{\pi} \int_{-1}^{+1} \frac{\cos xt}{\sqrt{(1-t^2)}}\, dt.$$

Prove that

$$J_0'' + \frac{1}{x} J_0' + J_0 = 0.$$

5. For any non-negative integral index *n,* the Bessel function $J_n(x)$ may be defined by

$$J_n(x) = \frac{x^n}{1 \cdot 3 \cdot 5 \ldots (2n-1)\,\pi} \int_{-1}^{+1} \cos xt\,(1-t^2)^{n-\frac{1}{2}}\, dt.$$

Prove that

(a) $$J_n'' + \frac{1}{x} J_n' + \left(1 - \frac{n^2}{x^2}\right) J_n = 0 \qquad (n \geqq 0),$$

(b) $$J_{n+1} = J_{n-1} - 2J_n' \qquad (n \geqq 1)$$

and $$J_1 = -J_0'.$$

Hints and Answers

4.2 THE INTEGRAL OF A CONTINUOUS FUNCTION OVER A REGION OF THE PLANE OR OF SPACE

4.2.1 The Double Integral (Domain Integral) as a Volume:. The first and most important generalization of the ordinary integral, like of the ordinary integral, is suggested by **geometrical intuition**. Let *R* be a closed region of the *xy*-plane, bounded - as we will assume all along—by one or more arcs of curves with continuously turning tangents and $z = f(x, y)$ a function which is continuous in *R.* In the first instance, we assume that *f* is non-negative and represent it by a surface in *xyz*-space vertically above the region *R.* We now wish to find (or, more precisely) to **define**, since we have not yet done so, the **volume V below the surface**. This has been done in detail for **rectangular regions** in Volume I, 10.6.1 and, moreover, the problem is so similar to that of the **ordinary integral** that we feel justified in mentioning it somewhat briefly here. The student will see at once that a natural way of arriving at this volume is to subdivide R into N sub-regions $R_1, R_2, \cdots, R_N$ with **sectionally smooth boundaries** and to find the l**argest** value M_i and the **smallest** value m_i of f in each region R_i. Denote the areas of the regions R_i by ΔR_i. Above each region R_i as base, we construct a cylinder of height M_i. This set of cylinders completely encloses the volume under the surface. Again, with each region R_i as base, we construct a cylinder of height m_i, and hence with volume $m_i\Delta R_i$; these cylinders lie completely within the volume under the surface. Hence

$$\sum_1^N m_i \Delta R_i \leq V \leq \sum_1^N M_i \Delta R_i.$$

We call these sums $\Sigma\Delta m_i\Delta R_i$ and $\Sigma M_i\Delta R_i$ **lower** and **upper sums**, respectively.

If we now make our subdivision finer and finer, so that the number N increases beyond all bounds, while the largest diameter of the regions R_i (i.e., the largest distance between two points of R_i) at the same time tends to zero, we see intuitively (and shall later prove rigorously) that the **upper** and **lower sums** must approach each other more and more closely, so that the volume V can be regarded as the common limit of the upper and lower sums as N tends to ∞.

Obviously, we can obtain the same limiting value if we take instead of m_i or M_i any number between m_i and M_i. e.g., $f(x_i,y_i)$, the value of the function at a point (x_i, y_i) in the region R_i.

4.2.2 The General Analytical Concept of the Integral: These concepts, which are suggested by geometry, must now be studied analytically and made more precise without direct reference to intuition. Accordingly, proceed as follows: Consider a closed region R with area ΔR and a function $f(x, y)$ which is defined and continuous everywhere in R, including the boundary. As before, we subdivide the region by sectionally smooth arcs, i.e., arcs which are given in a suitable co-ordinate system by an equation $y = \phi(x)$, where ϕ is a continuous function the derivative of which is continuous except for a finite number of jump discontinuities , into N sub-regions $R_1, R_2, \cdots, R_N$ with areas $\Delta R_1, \Delta R_2, \cdots, \Delta R_N$. Choose in R_i an arbitrary point (ξ_i, η_i), where the function has the value $f_i = f(\xi_i, \eta_i)$ and form the sum

$$V_N = \sum_1^N f_i \Delta R_i.$$

Then, the **fundamental theorem** is: If the number N increases beyond all bounds and at the same time the largest of the diameters of the sub-regions tends to zero, then V_N tends to a limit V. This limit is independent of the particular nature of the subdivision of the regions R and of the choice of the point (ξ_i, η_i) in R_i. We call the limit V the (**double**) **integral of the function $f(x, y)$ over the region R**: In symbols,

$$\iint_R f(x, y)\, dS.$$

Corollary: We obtain, the same limit if we take the sum only over those sub-regions R_i which lie entirely in the interior of R, *i.e.,* which have no points in common with the boundary of R.

This **existence theorem for the integral** of a continuous function must be proved in a purely analytical way. The proof, which is very similar to the corresponding proof for one variable, is given in A4.1.3.

We can refine this theorem further in a way which is useful for many purposes. In the subdivision into N sub-regions, it is not necessary to choose a value which is actually assumed by the function $f(x, y)$ at a definite point (ξ_i, η_i). of the corresponding sub-region; it is sufficient to choose values which differ from the values of the function $f(\xi_i, \eta_i)$ by quantities which tend **uniformly** to zero as the subdivision is made finer. In other words, instead of the values of the function $f(\xi_i, \eta_i)$, we can consider the quantities

$$f_i = f(\xi_i, \eta_i) + \epsilon_{i,N}, \text{where } |\epsilon_{i,N}| < \epsilon_N, \lim_{N\to\infty} \epsilon_N = 0.$$

(Hence, the number $\varepsilon_{i,N}$ is the difference between the value of the function at a point of the i-th subregion of the subdivision into N sub-regions and the quantity f_i with which we form the sum.) This theorem is almost trivial; in fact, since the numbers $\varepsilon_{i,N}$ tend *uniformly* to zero, the absolute value of the difference between the two sums

$$\sum_1^N f_i \Delta R_i \quad \text{and} \quad \sum_1^N (f_i + \epsilon_{i,N}) \Delta R_i$$

is less than $\varepsilon_N \Sigma \Delta R_i$ and can be made as small as we please by taking the number N sufficiently large. For example, if we have $f(x,y)=P(x,y)Q(x, y)$, we may take $f_i = P_i Q_i$, where P_i and Q_i are the maxima of P and Q in R, which are, in general, not assumed at the same point.

We shall now illustrate this concept of an integral by considering some **special subdivisions**. The simplest case is that in which R is a **rectangle** $a \le x \le b$, $c \le y \le d$ and the sub-regions R_i are also **rectangles**, formed by subdividing the x-interval into n and the y-interval into m equal parts of lengths

$$h = \frac{b-a}{n} \quad \text{and} \quad k = \frac{d-c}{m}.$$

We call the points of subdivision $x_0 = a, x_1, x_2, \cdots, x_n = b$ and $y_0 = c, y, y, \cdots, y_m = d$, respectively, and draw through these points parallels to the y-axis and the x-axis, respectively. We then have $N = nm$. All the sub-regions are rectangles with area $\Delta R_i = hk = \Delta x \Delta x$, if we set $h = \Delta x$, $k = \Delta y$. For the point (ξ_i, η_i), we can take any point in the corresponding rectangle and then form the sum

$$\sum_i f(\xi_i, \eta_i) \Delta x \Delta y$$

for all the rectangles of the subdivision. If we now let n and m increase simultaneously beyond all bounds, the sum will tend to the integral of the function f over the rectangle R.

These rectangles can also be characterized by two subscripts μ and ν, corresponding to the co-ordinates $x = a + \nu h$ and $y = c + \mu k$ of the **lower left-hand corner of the rectangle** in question. Here ν assumes integral values from 0 to $(n - 1)$ and μ from 0 to (m - 1). With this identification of the rectangles by the subscripts ν and μ, we may appropriately write the sum as a double sum

$$\sum_{\nu=0}^{n-1} \sum_{\mu=0}^{m-1} f(\xi_\nu, \eta_\mu)\Delta x \Delta y.$$

If we are to write the sum in this way, we must assume that the points (ξ_i, η_i) are chosen so as to lie on straight vertical or horizontal lines.

Even when R is not a rectangle, it is often convenient to subdivide the region into rectangular sub-regions R_i. In order to do so, we superimpose on the plane the **rectangular net** formed by the lines

$$x = \nu h \quad (\nu = 0, \pm 1, \pm 2, \ldots),$$
$$y = \mu k \quad (\mu = 0, \pm 1, \pm 2, \ldots),$$

where h and k are arbitrarily chosen numbers. Consider now all those rectangles of the subdivision which lie entirely within R. Call these rectangles R_i. Naturally, they do not completely cover the region; on the contrary, in addition to these rectangles, R also contains certain regions R_i adjacent to the boundary which are bounded **partly by lines** of the net and **partly by portions of the boundary** of R. However, by the corollary above, we can calculate the integral of the function over the region R by summing over the internal rectangles only and then passing to the limit.

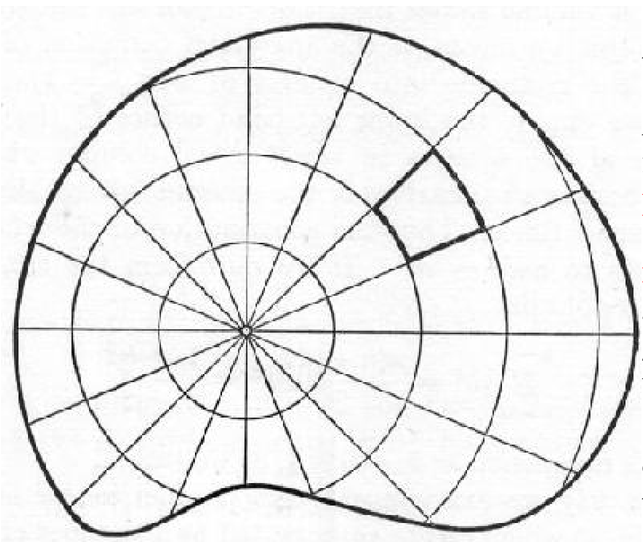

Fig. 3.—Subdivision by polar co-ordinate nets

Another type of subdivision, which is frequently applied, is the subdivision by a **polar co-ordinate net** (Fig. 3). Let the **origin** ***O*** of the polar co-ordinate system lie **in the interior** of our region. We subdivide the entire angle 2π into n parts of magnitude $\Delta\theta = 2\pi/n = h$ and also choose a second quantity $k = \Delta r$. We now draw the lines $\theta = \nu h$ ($\nu = 0, 1, 2, \ldots, n - 1$) through the origin as well as the concentric circles $r_\mu = \mu k$ ($\mu = 1, 2, \cdots$). Denote those which lie entirely inside R by R_i and their areas by ΔR_i. We can then regard the integral of the function $f(x, y)$ over the region R as the limit of the sum

$$\Sigma f(\xi_i, \eta_i)\Delta R_i$$

where (ξ_i, η_i) is a point chosen arbitrarily in R_i. The sum is taken over all the sub-regions R_i inside R and the passage to the limit involves letting h and k tend to zero simultaneously.

By **elementary geometry**, the area ΔR_i is given by the equation

$$\Delta R_i = \tfrac{1}{2}({r^2}_{\mu+1} - {r_\mu}^2)h = \tfrac{1}{2}(2\mu + 1)k^2 h,$$

if we assume that R_i lies in the ring bordered by the circles with radii μk and $(\mu + l)k$.

4.2.3. Examples:.The **simplest example** is the function $f(x, y) = 1$, when the limit of the sum is obviously independent of the mode of subdivision and is always equal to the area of the region R. Consequently, the integral of the function $f(x, y)$ over the region is also equal to this **area**. This might have been expected, because the integral is the **volume of the cylinder of unit height** with the base R.

As a further example, consider the integral of the function $f(x, y) = x$ over the **square** $0 \leq x \leq 1$, $0 \leq y \leq 1$. The intuitive interpretation of the integral as a volume shows that the value of our integral must be 1/2. We can verify this by means of the analytical definition of the integral. We subdivide the rectangle into squares with side $h = l/n$ and choose for the point (ξ_i, η_i) the lower left-hand corner of the small square. Then everyone of the squares in the vertical column, the left-hand side of which has the abscissa νh, contributes to the sum the amount νhI. This expression occurs n times. Thus, the contribution of the entire column of squares amounts to $n\nu h$i = νhI. If we now form the sum from ν = 0to $\nu = n - 1$, we obtain

$$\sum_{\nu=0}^{n-1} \nu h^2 = \frac{n(n-1)}{2} h^2 = \frac{1}{2} - \frac{h}{2}.$$

The limit of this expression as $h \to 0$ is 1/2, as we have already stated.

In a similar way, we can integrate the product xy or, more generally, any function $f(x, y)$ which can be represented as a product of a function of x and a function of y in the form $f(x, y) = \varphi(x)\psi(y)$, provided that the region of integration is a rectangle with sides parallel to the co-ordinate axes, say

$$a \leqq x \leqq b,$$
$$c \leqq y \leqq d.$$

We use the same subdivision of the rectangle as before and take for the value of the function in each sub-rectangle the value of the function at the lower left-hand corner. The integral is then the limit of the sum

$$hk \sum_{\nu=0}^{n-1} \sum_{\mu=0}^{m-1} \varphi(\nu h)\psi(\mu k),$$

which may also be written as the product of two sums

$$\left\{\sum_{\nu=0}^{n-1} h\,\varphi(\nu h)\right\} \left\{\sum_{\mu=0}^{m-1} k\,\psi(\mu k)\right\}.$$

However, in accordance with the definition of the ordinary integral, as $h \to 0$ and $k \to 0$, each of these factors tends to the integral of the corresponding function over the interval from a to b or from c to d, respectively. Thus, we obtain the **general rule**: If a function $f(x, y)$ can be represented as a product of two functions $\varphi(x)$ and $\psi(y)$, its double integral over a rectangle $a \leq x \leq b$, $c \leq y \leq d$ can be resolved into the product of two integrals:

$$\iint_R f(x, y)\,dx\,dy = \int_a^b \varphi(x)\,dx \cdot \int_c^d \psi(y)\,dy.$$

For example, by virtue of this rule and the **summation rule**, we can integrate any polynomial over a rectangle with sides parallel to the co-ordinate axes.

As a last example, we consider a case in which it is convenient to use a subdivision by the **polar co-ordinate net** instead of one by rectangles. Let the region *R* be the circle with unit radius and the origin as centre, given by *a*I+*y*I≤1, and let

$$f(x, y) = \sqrt{(1 - x^2 - y^2)};$$

in other words, we wish to find the volume of a **hemisphere of unit radius**.

We construct the **polar co-ordinate net** as before. From the sub-region, lying between the circles with radii $r_\mu = \mu k$ and $r_{\mu+1} = (\mu + 1)k$ and between the lines $\theta = \nu h$ and $\theta = (\nu + 1)h$ ($h = 2\pi/n$). we obtain the contribution

$$\frac{1}{2}\sqrt{1 - \left(\frac{r_{\mu+1} + r_\mu}{2}\right)^2}\,(r^2_{\mu+1} - r_\mu^2)h = \sqrt{1 - \rho_\mu^2}\,\rho_\mu kh,$$

where we have taken for the value of the function in the sub-region R_i the value which it assumes on an intermediate circle with the radius $\rho_\mu = (r_{\mu+1} + r_\mu)/2$. All the sub-regions which lie in the same ring make the same contribution and, since there are $n = 2\pi/h$ such regions, the contribution of the entire ring is

$$2\pi\sqrt{1 - \rho_\mu^2}\,\rho_\mu k.$$

Hence, the integral is the limit of the sum

$$\sum_{\mu=0}^{m-1} 2\pi\sqrt{1 - \rho_\mu^2}\,\rho_\mu k,$$

and, as we already know, this sum tends to the single integral

$$2\pi\int_0^1 r\sqrt{1 - r^2}\,dr = -\frac{2\pi}{3}\sqrt{(1 - r^2)^3}\,\Big|_0^1 = \frac{2\pi}{3};$$

hence, we obtain

$$\iint_R \sqrt{1 - x^2 - y^2}\,dS = \frac{2\pi}{3},$$

in agreement with the known formula for the volume of a sphere.

4.2.4. Notation. Extensions. Fundamental Rules: The rectangular subdivision of the region *R* is associated with the symbol for the double integral which has been in use since **Leibnitz**'s time. Starting with the symbol

$$\sum_{\nu=0}^{n-1}\sum_{\mu=0}^{m-1} f(\xi_\nu, \eta_\mu)\Delta x\Delta y$$

for the sum over rectangles, we indicate the passage to the limit from the sum to the integral by replacing the double summation sign by a double integral sign and writing the **symbol *dxdy*** instead of the **product** of the quantities Δx and Δy. Accordingly, the double integral is frequently written in the form

$$\int\int_R f(x, y)\,dx\,dy$$

instead of in the form

$$\int\int_R f(x, y)\,dS$$

in which the area of ΔR is replaced by the symbol dS. We again emphasize that the **symbol *dxdy*** does not mean a product, but merely refers symbolically to the passage to the limit of the above sums of nm terms *as* $n\to\infty$ and $m\to\infty$.

It is clear that in double integrals, just as in ordinary integrals of a single variable, the notation for the **integration variables** is immaterial, so that we could equally well have written

$$\int\int_R f(u, v)\,du\,dv \quad \text{or} \quad \int\int_R f(\xi, \eta)\,d\xi\,d\eta.$$

In introducing the **concept of integral**, we have seen that for a positive function $f(x, y)$ the integral represents the volume under the surface $z = f(x, y)$. However, in the analytical definition of the integral, it is quite unnecessary that the function $f\{x, y)$ should be positive everywhere; it may be negative or it may change sign, in which last case the surface intersects the region R. Thus, in the general case, the integral yields the **volume** in question with a **definite sign,** the sign being positive for surfaces or portions of surfaces which lie above the xy-plane.. If the entire surface, corresponding to the region R, consists of several such portions, the integral represents the sum of the corresponding volumes taken with their proper signs. In particular, a **double integral may vanish**, although the function under the integral sign does not vanish everywhere.

For double integrals, as for single integrals, the following **fundamental rules** hold, the proofs being a simple repetition of those in Volume I, 2.1.3. If c is a constant, then

$$\int\int_R cf(x, y)\,dS = c\int\int_R f(x, y)\,dS.$$

Also,

$$\iint_R (f(x, y) + \phi(x, y))\, dS = \iint_R f(x, y)\, dS + \iint_R \phi(x, y)\, dS,$$

that is: The inlegral of the sum of two functions is equal to the sum of their two integrals.
Finally, if the region R consists of two sub-regions R' and R'' which have at most portions of the boundary in common, then

$$\iint_R f(x, y)\, dS = \iint_{R'} f(x, y)\, dS + \iint_{R''} f(x, y)\, dS,$$

that is: When regions are joined, the corresponding integrals are added.

4.2.5 Integral Estimates and the Mean Value Theorem: As in the case of one independent variable, there are some very useful **estimation theorems** for double integrals. Since the proofs are practically the same as those of Volume I, 2.7.1 , we shall here be content with a statement of the facts.

If $f(x, y) \geq 0$ in R, then

$$\iint_R f(x, y)\, dS \geqq 0;$$

similarly, if $f(x, y) \leq 0$,

$$\iint_R f(x, y)\, dS \leqq 0.$$

This leads to the following result: If the inequality

$$f(x, y) \geqq \phi(x, y)$$

holds everywhere in R, then

$$\iint_R f(x, y)\, dS \geqq \iint_R \phi(x, y)\, dS.$$

A direct application of this theorem gives the relations

$$\iint_R f(x, y)\, dS \leqq \iint_R |f(x, y)|\, dS$$

and

$$\iint_R f(x, y)\, dS \geqq -\iint_R |f(x, y)|\, dS.$$

We can also combine these two inequalities in a single formula:

$$\left| \iint_R f(x, y)\, dS \right| \leqq \iint_R \left| f(x, y) \right| dS.$$

If m is the lower bound and *M* the upper bound of the values of the function *f*(*x, y*) in *R,* then

$$m\Delta R \leqq \iint f(x, y)\, dS \leqq M\Delta R,$$

where Δ*R* is the area of the region *R*. The integral can then be expressed in the form

$$\iint_R f(x, y)\, dS = \mu \Delta R,$$

where μ is a number intermediate between *m* and *M,* the exact value of which cannot, in general, be specified more exactly.

Just as in the case of continuous functions of one variable, we can state that the value μ is certainly assumed at **some point** of the region *R* by the **continuous function *f*(*x*, *y*).**

This form of the **estimation formula** we again call the **mean value theorem of the integral calculus**. Again, there applies the generalization: If *p*(*x, y*) is an arbitrary positive continuous function in *R,* then

$$\iint_R p(x, y) f(x, y)\, dS = \mu \iint_R p(x, y)\, dS,$$

where μ denotes a number between the largest and smallest values of *f*, which cannot be further specified.

As before, these integral estimates show that the integral varies continuously with the function. More precisely, if *f*(*x,y*) and ϕ(*x, y*) are two functions which satisfy the inequality

$$|f(x, y) - \phi(x, y)| < \epsilon,$$

where ε is a fixed positive number in the whole region *R* with area Δ*R*,, then the integrals

$$\iint_R f(x, y)\, dS \text{ and} \iint_R \phi(x, y)\, dS$$

differ by less than $\varepsilon\Delta R$, i.e., by less than a number which tends to zero with ε.

In the same way we see that the integral of a function varies continuously with the region. In fact, let the two regions *R′* and *R″* be obtained from one another by addition or removal of sections, the total area of which is less than ε and *f*(*x,y*), be a function, which is continuous in both regions and such that *f*(*x, y*) < *M,* where *M is* a fixed number. Then, the two integrals

$$\iint_{R'} f(x, y)\, dS \text{ and } \iint_{R''} f(x, y)\, dS$$

differ by less than $M\varepsilon$, i.e., by less than a number which tends to zero with ε. The proof of this fact follows at once from the last theorem of 4.2.2.

We can therefore calculate the **integral over a region *R*** as accurately as we please by taking it over a sub-region of R, the total area of which differs from. the area of R by a sufficiently small amount.For example, we can construct in the region R a polygon, the total area of which differs by as little as we please from the area of R. In particular, we may suppose this polygon to be bounded by lines parallel to the x- and y-axes, alternately, i.e., to be composed out of rectangles with sides parallel to the axes.

4.2.6. Integrals over Regions in Three and More Dimensions:.Every statement we have made for integrals over regions of the xy-plane can be extended without further complication or the introduction of new ideas to regions in three or more dimensions. For example, if we consider the case of the **integral over a three dimensional region** R, we need only subdivide this region R by means of a finite number of surfaces with continuously varying tangent planes into sub-regions which completely fill R and which we will denote by $R_1, R_2, \cdots, R_n$. If $f(x, y, z)$ is continuous in the closed region R and (ξ_i, η_i, ζ_i) denotes an arbitrary point in the region R, we form again the sum

$$\sum_{i=1}^{N} f(\xi_i, \eta_i, \zeta_i)\Delta R_i,$$

in which ΔR_i denotes the volume of the region R_i. The sum is taken over all the regions R_i or, if it is more convenient, only over those sub-regions which do not adjoin the boundary of R. If we now let the number of sub-regions increase beyond all bounds in such a way that the diameter of the largest of them tends to zero, we find again a limit independent of the particular mode of subdivision and of the choice of the intermediate points. We call this limit the **integral of *f*(*x*, *y*, z) over the region *R*** and denote it symbolically by

$$\iiint_R f(x, y, z)\,dV.$$

In particular, if we subdivide the region into rectangular regions with sides Δx, Δy, Δz, the volumes of the inner regions R_i will all have the same value $\Delta x\Delta y\Delta z$. As in 4.2.4, we indicate the possibility of this type of subdivision and the passage to the limit by introducing the symbolic notation

$$\iiint_R f(x, y, z)\,dx\,dy\,dz,$$

in addition to the above notation. All the facts which we have stated for double integrals remain valid for triple integrals apart from necessary changes in notation.

For regions of more than three dimensions, multiple integral can be defined in exactly the same way, once we have suitably defined the concept of volume for such regions. If, in the first instance, we restrict ourselves to rectangular regions and subdivide them into similarly oriented rectangular sub-regions, and moreover define the volume of a rectangle

$$a_1 \leqq x_1 \leqq a_1 + h_1, \quad a_2 \leqq x_2 \leqq a_2 + h_2, \ldots, \quad a_n \leqq x_n \leqq a_n + h_n,$$

as the product $h_1 h_2 \cdots h_n$, the definition of integral involves nothing new. We denote an **integral over the *n*-dimensional region** *R* by

$$\int\int \cdots \cdots \int_R f(x_1, x_2, \ldots, x_n)\, dx_1\, dx_2 \ldots dx_n.$$

For more general regions and more general subdivisions we must rely on the abstract definition of volume which we shall give in section A5.4.3(p. 387).

In the sequel, apart from in A4.3, we can confine ourselves to integrals in at most three dimensions.

4.2.7 Space Differentiation. Mass and Density: In the case of single integrals and functions of one variable, we obtain the **integrand** from the integral by a process of **differentiation**, taking the integral over an interval of length *h,* dividing by the length *h* and then letting *k* tend to zero. For functions of one variable, this fact represents the **fundamental link between the differential and integral calculus** and we interpreted it intuitively in terms of the concepts of total mass and density. For multiple integrals of functions of several variables, the same link exists; but here it is not so fundamental in character.

We consider the **multiple integral** (**domain integral**)

$$\int\int_B f(x, y)\, dS \quad \text{or} \quad \int\int\int_B f(x, y, z)\, dV$$

of a continuous function of two or more variables over a region *B* which contains a fixed point *P* with co-ordinates (x_0, y_0) - or (x_0, y_0, z_0) as the case may be - and the **content** ΔB.

The word **content** is used as a general word to include the idea of **length** in one dimension, **area** in two dimensions, **volume** in three dimensions, etc.

If we then divide the value of this integral by the **content** ΔB, it follows from the considerations of 4.2.5 that the quotient will be an intermediate value of the integrand, i.e., a number between the largest and smallest values of the integrand in the region. If we now let the diameter of the region *B* about the point *P* tend to zero, so that the content ΔB also tends to zero, this intermediate value of the function $f(x, y)$ - or $f(x, y, z)$ - must tend to the value of the function at the point *P.* Thus, the passage to the limit yields the relations

$$\lim_{\Delta B \to 0} \frac{1}{\Delta B} \int\int_B f(x, y)\, dS = f(x_0, y_0)$$

and

$$\lim_{\Delta B \to 0} \frac{1}{\Delta B} \int\int\int_B f(x, y, z)\, dV = f(x_0, y_0, z_0).$$

We call this limiting process, which corresponds to the differentiation described above for integrals with one independent variable, the **space differentiation** of the integral. Thus, we see that the space differentiation of a multiple integral yields the integrand.

This link enables us to interpret the relationship of the integrand to the integral in the case of several independent variables, as before, by means of the physical concepts of **density** and **total mass**. We think of a mass of any substance whatever as being distributed over a two- or three-dimensional region R in such a way that an arbitrarily small mass is contained in each sufficiently small sub-region. In order to define the specific mass or density at a point P, we first consider a neighbourhood B of the point P with content ΔB and divide the total mass in this neighbourhood by the content. We shall call the quotient the **mean density** or **average density** in this sub-region. If we now let the diameter of B tend to zero, we obtain, in the limit, from the average density in the region B the **density at the point *P*,** provided always that such a limit exists independently of the choice of the sequence of regions. If we denote this density by $\mu(x, y)$ - or by $\mu(x, y, z)$ - and assume that it is continuous, we see at once that the process described above is simply the space differentiation of the integral

$$\iint_R \mu(x, y)\, dS, \text{ or } \iiint_R \mu(x, y, z)\, dV,$$

taken over the entire region R. This integral therefore gives us the **total mass** of the substance of **density** μ in the region R.

What we have shown here *is* that the distribution given by the multiple integral has the same space-derivative as the mass-distribution originally given. It remains to be proved that this implies that the two distributions are actually identical; in other words, that the statement space differentiation gives the density μ can be satisfied by only one distribution of mass. The proof, which is not difficult, will be omitted. It closely resembles the proof of the **Heine-Borel Covering Theorem**.

From the physical point of view, such a representation of the mass of a substance is naturally an idealization. That this idealization is reasonable, i.e., that it approximates the actual situation with sufficient accuracy, is one of the assumptions of **physics**.

These ideas, moreover, retain their mathematical significance even when μ is not positive everywhere. Such negative densities and masses may also have a physical interpretation, e.g., in the study of the distribution of **electric charge**.

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